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Exact quantum theory of noninteracting electrons with time-dependent effective mass in a time-dependent magnetic field

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Abstract

We investigated the quantum states of a free electron with time-dependent effective mass subjected to a time-dependent magnetic field by solving the Schrödinger equation under the choice of Landau and symmetric gauges. Using the invariant operator and unitary transformation methods together, we derived exact wavefunctions of the system. The wavefunctions rely on the solution of associated classical dynamical systems. We confirmed that the quantum analysis of the system under the two gauges coincides mutually.

1. Introduction

The study of quantum systems with time-dependent masses [1–4] as well as with position-dependent masses [5–9] has been a widespread subject and may be applied to various branches of physics. For instance, Colegrave *et al* used them to investigate the field intensities in a Fabry–Perot cavity [10] and they suggested possible applications to solid state physics and quantum field theory [11]. Remaud *et al* [12] found that a varying mass parameter offers a means of simulating an input or removal of energy from the system. They remark that, if energy is supplied to an oscillator in a periodic cycle of time, the resulting dynamics can be described by a function of periodic mass. The wavefunctions, uncertainty relations and propagators are obtained for the harmonic oscillator with an exponentially decaying mass [13]. The exact quantum theory of a pendulum with a linearly decreasing mass was constructed [14]. In any system, the charged particles, such as electrons or holes, may interact with the circumference or various excitations such as temperature [15], pressure [16], stress [17] and energy [18], resulting in modifications to the effective masses. If the environment changes as time goes by, the effective masses may naturally vary, depending on time. When the external field changes randomly, the electron effective mass in the heterojunctions and solid solutions may be varied in a random fashion according to the fluctuation of the composition of the system [19].

One of the most powerful methods to solve the time-dependent quantum systems is the invariant operator method that was introduced initially by Lewis [20, 21]. We will use the invariant operator and unitary transformation methods together in order to investigate the exact quantum state of the system. The key idea in solving the harmonic oscillator with time-dependent mass is that the solution (wavefunction) of the Schrödinger equation for a time-dependent Hamiltonian system is the same as the eigenstate of the corresponding invariant operator, except for some time-dependent phase factor [22].

The quantum state of an electron moving in a two-dimensional plane with constant effective mass has been investigated in [23]. We will do the same for the system that replaced the ordinary effective mass with a time-dependent effective mass. Although many actual dynamical systems have been solved approximately using perturbation theory [24–26], we will confine our concern with the investigation of the exact quantum theory of the system. In most of the cases, the evolution of the time-dependent systems are periodic. Our discussion in this paper can be applied to quadratic Hamiltonian systems with random time dependence as well as time-periodic ones. Hagedorn *et al* [27] studied the relation between the classical and quantum motions associated with time-dependent quadratic Hamiltonians. The wavefunctions rely on the solution of the associated classical dynamical systems.

2. Quantum electronic structure with the choice of Landau gauge

We consider a free electron subject to a time-dependent vector potential $\boldsymbol{A}(t)$ in a two-dimensional plane with a time-dependent effective mass, $m^*(t)$. This system can be described by the following Hamiltonian:

$$\hat{H} = \frac{1}{2m^*(t)}[\hat{\boldsymbol{p}} - e\boldsymbol{A}(t)]^2, \tag{2.1}$$

where $-e$ is the charge of the electron. The magnetic field can be written in terms of $\boldsymbol{A}(t)$ as

$$\boldsymbol{B}(t) = \boldsymbol{\nabla} \times \boldsymbol{A}(t). \tag{2.2}$$

We are free to choose the gauge, since the gauge transformation has no effect on any physical result. With the choice of Landau gauge, we can write the vector potential as

$$\boldsymbol{A}(t) = (0, B(t)\hat{x}, 0). \tag{2.3}$$

Then, equation (2.1) becomes the Hamiltonian of the one-dimensional time-dependent harmonic oscillator [23]:

$$\hat{H} = -\frac{\hbar^2}{2m^*(t)}\left(\frac{\mathrm{d}^2}{\mathrm{d}\hat{x}^2} + \frac{\mathrm{d}^2}{\mathrm{d}\hat{y}^2}\right) + \frac{1}{2}m^*(t)\omega_c^2(t)[\hat{x} - \Delta(t)]^2, \tag{2.4}$$

where $\Delta(t)$ and $\omega_c(t)$ are given by

$$\Delta(t) = \frac{\hbar}{eB(t)}k_q, \tag{2.5}$$

$$\omega_c(t) = \frac{eB(t)}{m^*(t)}. \tag{2.6}$$

In equation (2.5), k_q is the wavenumber in the $\hat{y}$ direction. We impose periodic boundary conditions along the $\hat{y}$ direction over a distance L:

$$\psi_n(\hat{x}, \hat{y}, t) = \psi_n(\hat{x}, \hat{y} + L, t). \tag{2.7}$$

Then, k_q is given by

$$k_q = \frac{2\pi q}{L}, \qquad q = 0, \pm 1, \pm 2, \ldots. \tag{2.8}$$

The Hamiltonian, equation (2.4), can be separated into $\hat{x}$ and $\hat{y}$ components as

$$\hat{H} = \hat{H}_x + \hat{H}_y, \tag{2.9}$$

where

$$\hat{H}_x = -\frac{\hbar^2}{2m^*(t)}\frac{\mathrm{d}^2}{\mathrm{d}\hat{x}^2} + \frac{1}{2}m^*(t)\omega_c^2(t)[\hat{x} - \Delta(t)]^2, \tag{2.10}$$

$$\hat{H}_y = -\frac{\hbar^2}{2m^*(t)}\frac{\mathrm{d}^2}{\mathrm{d}\hat{y}^2}. \tag{2.11}$$

Equation (2.10) is the same as that of a time-dependent harmonic oscillator centred at $\Delta(t)$, while equation (2.11) is the same as that of a time-dependent free particle.

We can write the whole Schrödinger equation of the system as

$$\mathrm{i}\hbar\frac{\partial\psi_n(\hat{x}, \hat{y}, t)}{\partial t} = \hat{H}\psi_n(\hat{x}, \hat{y}, t). \tag{2.12}$$

The wavefunctions must be separable into $\hat{x}$ and $\hat{y}$ components. We therefore suppose that the wavefunctions can be expressed in the form

$$\psi_n(\hat{x}, \hat{y}, t) = u_n(\hat{x}, t)v(\hat{y}). \tag{2.13}$$

Note that, in equation (2.13), we tied the time variable to the coordinate $\hat{x}$. By substituting equations (2.9) and (2.13) into equation (2.12), we find that

$$\begin{aligned}\frac{\hbar^2}{2v(\hat{y})}\frac{\mathrm{d}^2v(\hat{y})}{\mathrm{d}\hat{y}^2} &= -\mathrm{i}\hbar\frac{m^*(t)}{u_n(\hat{x}, t)}\frac{\partial u_n(\hat{x}, t)}{\partial t} - \frac{\hbar^2}{2u_n(\hat{x}, t)}\frac{\mathrm{d}^2u_n(\hat{x}, t)}{\mathrm{d}\hat{x}^2}\\ &\quad + \tfrac{1}{2}m^{*2}(t)\omega_c^2(t)[\hat{x} - \Delta(t)]^2 = -\mathcal{E}_q,\end{aligned} \tag{2.14}$$

where $\mathcal{E}_q$ is the separation constant with the dimension of energy times mass. Then, equation (2.14) can be reexpressed with $\hat{x}$ and $\hat{y}$ components separately:

$$\mathrm{i}\hbar\frac{\partial u_n(\hat{x}, t)}{\partial t} = H_x'(\hat{x}, \hat{p}_x, t)u_n(\hat{x}, t), \tag{2.15}$$

$$\frac{\mathrm{d}^2v(\hat{y})}{\mathrm{d}\hat{y}^2} + \frac{2\mathcal{E}_q}{\hbar^2}v(\hat{y}) = 0, \tag{2.16}$$

where

$$\hat{H}_x'(\hat{x}, \hat{p}_x, t) = \frac{\hat{p}_x^2}{2m^*(t)} + \frac{1}{2}m^*(t)\omega_c^2(t)[\hat{x}^2 - 2\Delta(t)\hat{x} + \Delta^2(t)] + \frac{\mathcal{E}_q}{m^*(t)}. \tag{2.17}$$

The normalized solution of equation (2.16) is given by

$$v(\hat{y}) = \frac{1}{\sqrt{L}}\exp(\mathrm{i}k_q\hat{y}) \tag{2.18}$$

and the relation between k_q and E_q is

$$k_q = \sqrt{\frac{2\mathcal{E}_q}{\hbar^2}}. \tag{2.19}$$

To solve the $\hat{x}$ component of the Schrödinger equation, we introduce the invariant operator, $\hat{I}_x(t)$, that can be written as

$$\begin{aligned}\hat{I}_x(t) &= \alpha(t)[\hat{p}_x - p_{x,\mathrm{p}}(t)]^2 + \beta(t)\{[\hat{x} - x_\mathrm{p}(t)][\hat{p}_x - p_{x,\mathrm{p}}(t)] + [\hat{p}_x - p_{x,\mathrm{p}}(t)][\hat{x} - x_\mathrm{p}(t)]\}\\ &\quad + \gamma(t)[\hat{x} - x_\mathrm{p}(t)]^2,\end{aligned} \tag{2.20}$$

where the time-dependent coefficients $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ must be determined afterward and $x_{\mathrm{p}}(t)$ and $p_{x,\mathrm{p}}(t)$ are particular solutions of the classical equation of motion for the $\hat{x}$ component in coordinate and momentum space, respectively. They satisfy the following relations:

$$\dot{x}_{\mathrm{p}}(t) = \frac{1}{m^*(t)} p_{x,\mathrm{p}}(t), \tag{2.21}$$

$$\dot{p}_{x,\mathrm{p}}(t) = -m^*(t)\omega_c^2(t)[x_{\mathrm{p}}(t) - \Delta(t)]. \tag{2.22}$$

Because of its definition, the invariant operator must satisfy the relation

$$\frac{\mathrm{d}\hat{I}_x(t)}{\mathrm{d}t} = \frac{\partial \hat{I}_x(t)}{\partial t} + \frac{1}{\mathrm{i}\hbar}[\hat{I}_x(t), \hat{H}'_x] = 0. \tag{2.23}$$

By substituting equations (2.17) and (2.20) into (2.23), we can derive $\alpha(t) - \gamma(t)$ as

$$\alpha(t) = c_1\rho_+^2(t) + c_2\rho_-^2(t), \tag{2.24}$$

$$\beta(t) = -m^*(t)[c_1\rho_+(t)\dot{\rho}_+(t) + c_2\rho_-(t)\dot{\rho}_-(t)], \tag{2.25}$$

$$\gamma(t) = m^{*2}(t)[c_1\dot{\rho}_+^2(t) + c_2\dot{\rho}_-^2(t)], \tag{2.26}$$

where $\rho_\pm(t)$ are the two independent classical solutions of the following differential equation:

$$\ddot{\rho}_\pm(t) + \frac{\dot{m}^*(t)}{m^*(t)}\dot{\rho}_\pm(t) + \omega_c^2(t)\rho_\pm(t) = 0. \tag{2.27}$$

To derive the eigenstate of the invariant operator, we introduce the unitary operator $\hat{U}$ as

$$\hat{U} = \hat{U}_3\hat{U}_2\hat{U}_1, \tag{2.28}$$

where

$$\hat{U}_1 = \exp\left(\frac{\mathrm{i}}{\hbar}x_{\mathrm{p}}\hat{p}_x\right)\exp\left(-\frac{\mathrm{i}}{\hbar}p_{x,\mathrm{p}}\hat{x}\right), \tag{2.29}$$

$$\hat{U}_2 = \exp\left(\frac{\mathrm{i}\beta}{2\alpha\hbar}\hat{x}^2\right), \tag{2.30}$$

$$\hat{U}_3 = \exp\left\{\frac{\mathrm{i}}{4\hbar}(\hat{x}\hat{p}_x + \hat{p}_x\hat{x})\ln[2\alpha m^*(0)]\right\}. \tag{2.31}$$

Then, we can transformation equation (2.20) with (2.28) into a very simple form:

$$\hat{I}'_x = \hat{U}\hat{I}_x\hat{U}^\dagger = \frac{\hat{p}_x^2}{2m^*(0)} + \frac{1}{2}m^*(0)\omega^2\hat{x}^2, \tag{2.32}$$

where ω^2 is a constant given by the relation

$$\omega^2 = 4(\alpha\gamma - \beta^2) = 4c_1c_2m^{*2}(t)(\rho_+\dot{\rho}_- - \dot{\rho}_+\rho_-)^2. \tag{2.33}$$

The eigenvalue equation of $\hat{I}'_x$ can be written as

$$\hat{I}'_x\phi'_n(\hat{x}, t) = \lambda_n\phi'_n(\hat{x}, t). \tag{2.34}$$

Since equation (2.32) is just the same as the Hamiltonian of a simple harmonic oscillator, we can easily identify $\phi'_n(\hat{x}, t)$ as

$$\phi'_n(\hat{x}, t) = \sqrt[4]{\frac{m^*(0)\omega}{\hbar\pi}}\frac{1}{\sqrt{2^n n!}}H_n\left(\sqrt{\frac{m^*(0)\omega}{\hbar}}\hat{x}\right)\exp\left(-\frac{m^*(0)\omega}{2\hbar}\hat{x}^2\right), \tag{2.35}$$

where H_n is an nth-order Hermite polynomial and λ_n is

$$\lambda_n = \hbar\omega(n + \tfrac{1}{2}), \qquad n = 0, 1, 2, \ldots. \tag{2.36}$$

The eigenstate of an untransformed invariant operator, $\hat{I}_x$, can be derived from

$$\phi_n(\hat{x}, t) = \hat{U}^\dagger \phi'_n(\hat{x}, t). \tag{2.37}$$

Making use of equation (2.28) with (2.35), we can calculate equation (2.37) as

$$\phi_n(\hat{x}, t) = \sqrt[4]{\frac{\omega}{2\alpha\hbar\pi}} \frac{1}{\sqrt{2^n n!}} H_n\left(\sqrt{\frac{\omega}{2\alpha\hbar}}(\hat{x} - x_\mathrm{p})\right) \exp\left(\frac{\mathrm{i}}{\hbar} p_{x,\mathrm{p}}\hat{x}\right) \times \exp\left[-\frac{1}{2\alpha\hbar}\left(\frac{\omega}{2} + \mathrm{i}\beta\right)(\hat{x} - x_\mathrm{p})^2\right]. \tag{2.38}$$

The wavefunctions of the $\hat{x}$ component, $u_n(\hat{x}, t)$, differs from the eigenstate of the invariant operator only by some time-dependent phase factor, $\delta_n(t)$ [22]:

$$u_n(\hat{x}, t) = \phi_n(\hat{x}, t) \exp[\mathrm{i}\delta_n(t)]. \tag{2.39}$$

The phase factor can be derived by substituting equation (2.39) into (2.15) as

$$\delta_n(t) = -\omega\left(n + \frac{1}{2}\right)\int_0^t \frac{1}{2\alpha(t')m^*(t')}\,\mathrm{d}t' - \frac{1}{\hbar}\int_0^t H'_{x,\mathrm{p}}(x_\mathrm{p}(t'), p_{x,\mathrm{p}}(t'), t')\,\mathrm{d}t', \tag{2.40}$$

where

$$H'_{x,\mathrm{p}}(x_\mathrm{p}(t), p_{x,\mathrm{p}}(t), t) = \frac{p_{x,\mathrm{p}}^2(t)}{2m^*(t)} + \frac{1}{2}m^*(t)\omega_c^2(t)[x_\mathrm{p}^2(t) - 2\Delta(t)x_\mathrm{p}(t) + \Delta^2(t)] + \frac{\mathcal{E}_q}{m^*(t)}. \tag{2.41}$$

Thus, the full wavefunctions can be described exactly in terms of equations (2.13), (2.18) and (2.39) with (2.38) and (2.40).

Now, let us investigate a special case: that described by a constant angular frequency Ω so that the motion is periodic:

$$\omega_c(t) = \Omega(\equiv \text{constant}). \tag{2.42}$$

The time-dependent effective mass is given by

$$m^*(t) = m + m_0\mathrm{e}^{-\kappa t} \tag{2.43}$$

where m, m_0 and κ are real constants with the condition $m \gg m_0$. Then, equation (2.27) becomes

$$\ddot{\rho}_\pm(t) - \frac{m_0\kappa}{m}\mathrm{e}^{-\kappa t}\dot{\rho}_\pm(t) + \Omega^2\rho_\pm(t) \simeq 0. \tag{2.44}$$

The classical solutions of equation (2.44) can be derived as

$$\rho_+(t) = \mathrm{e}^{\mathrm{i}\Omega t}\,{}_1F_1\left(\frac{\mathrm{i}\Omega}{\kappa}, 1 + \frac{2\mathrm{i}\Omega}{\kappa}, -\frac{m_0}{m}\mathrm{e}^{-\kappa t}\right), \tag{2.45}$$

$$\rho_-(t) = \mathrm{e}^{\mathrm{i}\Omega t}\,{}_1F_1\left(-\frac{\mathrm{i}\Omega}{\kappa}, 1 - \frac{2\mathrm{i}\Omega}{\kappa}, -\frac{m_0}{m}\mathrm{e}^{-\kappa t}\right), \tag{2.46}$$

where ${}_1F_1(a; b; Z)$ is the Kummer confluent hypergeometric function [28]. In terms of equations (2.45) and (2.46), the wavefunction equation (2.39) can be explicitly represented.

3. Quantum state of electrons with the choice of symmetric gauge

None of the gauge transformations will affect the physical quantities. We will do the same thing as in the previous section by choosing a symmetric gauge which can be expressed by the following vector potential [23]:

$$\boldsymbol{A}(t) = \left(-\frac{\boldsymbol{B}(t)}{2}\hat{y}, \frac{\boldsymbol{B}(t)}{2}\hat{x}, 0\right). \tag{3.1}$$

The Hamiltonian corresponding to equation (3.1) is

$$\begin{aligned}\hat{\mathcal{H}}(\hat{x}, \hat{p}_x, \hat{y}, \hat{p}_y, t) &= \frac{1}{2m^*(t)}\left[\left(\hat{p}_x + \frac{eB(t)}{2}\hat{y}\right)^2 + \left(\hat{p}_y - \frac{eB(t)}{2}\hat{x}\right)^2 + \hat{p}_z^2\right] \\ &= -\frac{\hbar^2}{2m^*(t)}\left(\frac{\mathrm{d}^2}{\mathrm{d}\hat{x}^2} + \frac{\mathrm{d}^2}{\mathrm{d}\hat{y}^2} + \frac{\mathrm{d}^2}{\mathrm{d}\hat{z}^2}\right) + \frac{1}{8}m^*(t)\omega_c^2(t)(\hat{x}^2 + \hat{y}^2) \\ &\quad - \frac{1}{2}\omega_c(t)(\hat{x}\hat{p}_y - \hat{y}\hat{p}_x).\end{aligned} \tag{3.2}$$

The Schrödinger equation related to $\hat{\mathcal{H}}$ can be written as

$$\mathrm{i}\hbar\frac{\partial\psi_{n_x,n_y}(\hat{x}, \hat{y}, \hat{z}, t)}{\partial t} = \hat{\mathcal{H}}\psi_{n_x,n_y}(\hat{x}, \hat{y}, \hat{z}, t). \tag{3.3}$$

We can transform equation (3.3) with some unitary operator $\hat{V}_0$:

$$\hat{\mathcal{H}}' = \hat{V}_0^{-1}\hat{\mathcal{H}}\hat{V}_0 - \mathrm{i}\hbar\hat{V}_0^{-1}\frac{\partial\hat{V}_0}{\partial t}. \tag{3.4}$$

We choose $\hat{V}_0$ in the form

$$\hat{V}_0 = \exp\left[-\frac{\mathrm{i}\varphi}{2\hbar}(\hat{y}\hat{p}_x - \hat{x}\hat{p}_y)\right], \tag{3.5}$$

where φ is given by

$$\varphi = \int^t \omega_c(t')\,\mathrm{d}t' + \delta. \tag{3.6}$$

In the above equation, δ is the integral constant. Then, the transformed Hamiltonian can be calculated as

$$\hat{\mathcal{H}}' = \frac{1}{2m^*(t)}(\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2) + \frac{1}{8}m^*(t)\omega_c^2(t)(\hat{x}^2 + \hat{y}^2). \tag{3.7}$$

Let us write the Schrödinger equation related to $\hat{\mathcal{H}}'$ as

$$\mathrm{i}\hbar\frac{\partial\psi'_{n_x,n_y}(\hat{x}, \hat{y}, \hat{z}, t)}{\partial t} = \hat{\mathcal{H}}'\psi'_{n_x,n_y}(\hat{x}, \hat{y}, \hat{z}, t). \tag{3.8}$$

The solutions of the above equation are derived by separating variables:

$$\psi'_{n_x,n_y}(\hat{x}, \hat{y}, \hat{z}, t) = u'_{n_x,n_y}(\hat{x}, \hat{y}, t)v'(\hat{z}). \tag{3.9}$$

Then, using the same method as in the previous section, we can separate variables as

$$\mathrm{i}\hbar\frac{\partial u'_{n_x,n_y}(\hat{x}, \hat{y}, t)}{\partial t} = \hat{\mathcal{H}}'_{xy}(\hat{x}, \hat{p}_x, \hat{y}, \hat{p}_y, t)u'_{n_x,n_y}(\hat{x}, \hat{y}, t), \tag{3.10}$$

$$\frac{\mathrm{d}^2v'(\hat{z})}{\mathrm{d}\hat{z}^2} + \frac{2\mathcal{E}'_q}{\hbar^2}v'(\hat{z}) = 0, \tag{3.11}$$

where $\hat{\mathcal{H}}'_{xy}(\hat{x}, \hat{p}_x, \hat{y}, \hat{p}_y, t)$ is given by

$$\hat{\mathcal{H}}'_{xy}(\hat{x}, \hat{p}_x, \hat{y}, \hat{p}_y, t) = \frac{1}{2m^*(t)}(\hat{p}_x^2 + \hat{p}_y^2) + \frac{1}{8}m^*(t)\omega_c^2(t)(\hat{x}^2 + \hat{y}^2) + \frac{\mathcal{E}'_q}{m^*(t)}. \tag{3.12}$$

The corresponding solution of equation (3.11) can be written as

$$v'(\hat{z}) = \frac{1}{\sqrt{L}} \exp(\mathrm{i}k'_q\hat{z}). \tag{3.13}$$

where k'_q is given by

$$k'_q = \sqrt{\frac{2\mathcal{E}'_q}{\hbar^2}}. \tag{3.14}$$

Let us denote the invariant operator related to equation (3.12) as $\hat{I}_{xy}(t)$, and assume its trial form as

$$\hat{I}_{xy}(t) = \alpha_1(t)\hat{p}_x^2 + \beta_1(t)(\hat{x}\hat{p}_x + \hat{p}_x\hat{x}) + \gamma_1(t)\hat{x}^2 + \alpha_2(t)\hat{p}_y^2 + \beta_2(t)(\hat{y}\hat{p}_y + \hat{p}_y\hat{y}) + \gamma_2(t)\hat{y}^2, \tag{3.15}$$

where the coefficients $\alpha_i(t)$, $\beta_i(t)$ and $\gamma_i(t)$ $(i = 1, 2)$ must be determined afterward. Then, it satisfies the relation given by

$$\frac{\mathrm{d}\hat{I}_{xy}(t)}{\mathrm{d}t} = \frac{\partial \hat{I}_{xy}(t)}{\partial t} + \frac{1}{\mathrm{i}\hbar}[\hat{I}_{xy}(t), \hat{\mathcal{H}}'_{xy}] = 0. \tag{3.16}$$

By substituting equations (3.12) and (3.15) into (3.16), we can easily derive $\alpha_i(t) - \gamma_i(t)$ as

$$\alpha_1(t) = c_{x1}\rho_{x+}^2(t) + c_{x2}\rho_{x-}^2(t), \tag{3.17}$$

$$\beta_1(t) = -m^*(t)[c_{x1}\rho_{x+}(t)\dot{\rho}_{x+}(t) + c_{x2}\rho_{x-}(t)\dot{\rho}_{x-}(t)], \tag{3.18}$$

$$\gamma_1(t) = m^{*2}(t)[c_{x1}\dot{\rho}_{x+}^2(t) + c_{x2}\dot{\rho}_{x-}^2(t)], \tag{3.19}$$

$$\alpha_2(t) = c_{y1}\rho_{y+}^2(t) + c_{y2}\rho_{y-}^2(t), \tag{3.20}$$

$$\beta_2(t) = -m^*(t)[c_{y1}\rho_{y+}(t)\dot{\rho}_{y+}(t) + c_{y2}\rho_{y-}(t)\dot{\rho}_{y-}(t)], \tag{3.21}$$

$$\gamma_2(t) = m^{*2}(t)[c_{y1}\dot{\rho}_{y+}^2(t) + c_{y2}\dot{\rho}_{y-}^2(t)], \tag{3.22}$$

where $\rho_{x\pm}(t)$ and $\rho_{y\pm}(t)$ are the two independent real classical solutions of the following differential equation:

$$\ddot{\rho}_{x\pm}(t) + \frac{\dot{m}^*(t)}{m^*(t)}\dot{\rho}_{x\pm}(t) + \frac{\omega_c^2(t)}{4}\rho_{x\pm}(t) = 0, \tag{3.23}$$

$$\ddot{\rho}_{y\pm}(t) + \frac{\dot{m}^*(t)}{m^*(t)}\dot{\rho}_{y\pm}(t) + \frac{\omega_c^2(t)}{4}\rho_{y\pm}(t) = 0. \tag{3.24}$$

To derive the eigenstate of the invariant operator, let us transform equation (3.15) as

$$\hat{I}'_{xy} = \hat{V}\hat{I}_{xy}\hat{V}^\dagger, \tag{3.25}$$

where the unitary operator $\hat{V}$ is given by

$$\hat{V} = \hat{V}_2\hat{V}_1, \tag{3.26}$$

with

$$\hat{V}_1 = \exp\left(\frac{\mathrm{i}\beta_1}{2\alpha_1\hbar}\hat{x}^2\right)\exp\left(\frac{\mathrm{i}\beta_2}{2\alpha_2\hbar}\hat{y}^2\right), \tag{3.27}$$

$$\hat{V}_2 = \exp\left\{\frac{\mathrm{i}}{4\hbar}(\hat{x}\hat{p}_x + \hat{p}_x\hat{x})\ln[2\alpha_1 m^*(0)]\right\}\exp\left\{\frac{\mathrm{i}}{4\hbar}(\hat{y}\hat{p}_y + \hat{p}_y\hat{y})\ln[2\alpha_2 m^*(0)]\right\}. \tag{3.28}$$

Then, the transformed invariant operator can be written simply as

$$\hat{I}'_{xy} = \frac{1}{2m^*(0)}(\hat{p}_x^2 + \hat{p}_y^2) + \frac{1}{2}m^*(0)(\omega_1^2\hat{x}^2 + \omega_2^2\hat{y}^2), \tag{3.29}$$

where ω_1^2 and ω_2^2 are constants given by the relation

$$\omega_1^2 = 4(\alpha_1\gamma_1 - \beta_1^2) = 4c_{x1}c_{x2}m^{*2}(t)(\rho_{x+}\dot{\rho}_{x-} - \dot{\rho}_{x+}\rho_{x-})^2, \tag{3.30}$$

$$\omega_2^2 = 4(\alpha_2\gamma_2 - \beta_2^2) = 4c_{y1}c_{y2}m^{*2}(t)(\rho_{y+}\dot{\rho}_{y-} - \dot{\rho}_{y+}\rho_{y-})^2. \tag{3.31}$$

The eigenvalue equation of $\hat{I}'_{xy}$ can be written as

$$\hat{I}'_{xy}\phi'_{n_x,n_y}(\hat{x},\hat{y},t) = \lambda_{n_x,n_y}\phi'_{n_x,n_y}(\hat{x},\hat{y},t). \tag{3.32}$$

Since equation (3.29) is just the same as the Hamiltonian of a simple harmonic oscillator, we can easily identify $\phi'_{n_x,n_y}(\hat{x},\hat{y},t)$ as

$$\phi'_{n_x,n_y}(\hat{x},\hat{y},t) = \sqrt{\frac{m^*(0)(\omega_1\omega_2)^{1/2}}{\hbar\pi}}\frac{1}{\sqrt{2^{n_x+n_y}n_x!n_y!}}H_{n_x}\left(\sqrt{\frac{m^*(0)\omega_1}{\hbar}}\hat{x}\right) \times H_{n_y}\left(\sqrt{\frac{m^*(0)\omega_2}{\hbar}}\hat{y}\right)\exp\left(-\frac{m^*(0)}{2\hbar}(\omega_1\hat{x}^2 + \omega_2\hat{y}^2)\right) \tag{3.33}$$

and λ_{n_x,n_y} as

$$\lambda_{n_x,n_y} = \hbar\omega_1(n_x + \tfrac{1}{2}) + \hbar\omega_2(n_y + \tfrac{1}{2}), \qquad n_{x(y)} = 0, 1, 2, \ldots. \tag{3.34}$$

The eigenstate of the untransformed invariant operator, $\hat{I}_{xy}$, can be derived from

$$\phi_{n_x,n_y}(\hat{x},\hat{y},t) = \hat{V}^\dagger\phi'_{n_x,n_y}(\hat{x},\hat{y},t). \tag{3.35}$$

Using equation (3.26), the above equation can be calculated as

$$\phi_{n_x,n_y}(\hat{x},\hat{y},t) = \sqrt{\frac{(\omega_1\omega_2)^{1/2}}{2\hbar\pi(\alpha_1\alpha_2)^{1/2}}}\frac{1}{\sqrt{2^{n_x+n_y}n_x!n_y!}}H_{n_x}\left(\sqrt{\frac{\omega_1}{2\alpha_1\hbar}}\hat{x}\right)H_{n_y}\left(\sqrt{\frac{\omega_2}{2\alpha_2\hbar}}\hat{y}\right) \times \exp\left[-\frac{1}{2\alpha_1\hbar}\left(\frac{\omega_1}{2} + \mathrm{i}\beta_1\right)\hat{x}^2 - \frac{1}{2\alpha_2\hbar}\left(\frac{\omega_2}{2} + \mathrm{i}\beta_2\right)\hat{y}^2\right]. \tag{3.36}$$

The wavefunctions of the $\hat{x}$ and $\hat{y}$ components, $u'_{n_x,n_y}(\hat{x},\hat{y},t)$, differ from the eigenstates of the invariant operator only by some time-dependent phase factor, $\delta_{n_x,n_y}(t)$ [22]:

$$u'_{n_x,n_y}(\hat{x},\hat{y},t) = \phi_{n_x,n_y}(\hat{x},\hat{y},t)\exp[\mathrm{i}\delta_{n_x,n_y}(t)]. \tag{3.37}$$

By substituting equation (3.37) into (3.10), we can easily derive the phase factor as

$$\delta_{n_x,n_y}(t) = -\omega_1\left(n_x + \frac{1}{2}\right)\int_0^t \frac{1}{2\alpha_1(t')m^*(t')}\,\mathrm{d}t' - \omega_2\left(n_y + \frac{1}{2}\right)\int_0^t \frac{1}{2\alpha_2(t')m^*(t')}\,\mathrm{d}t'. \tag{3.38}$$

The wavefunctions of the system in a symmetric gauge can be calculated from

$$\psi_{n_x,n_y}(\hat{x},\hat{y},\hat{z},t) = \hat{V}_0\psi'_{n_x,n_y}(\hat{x},\hat{y},\hat{z},t). \tag{3.39}$$

Using equations (3.5) and (3.9) with (3.13) and (3.37), the above equation can be written as

$$\psi_{n_x,n_y}(\hat{x},\hat{y},\hat{z},t) = \phi_{n_x,n_y}(\hat{X},\hat{Y},t)\exp[\mathrm{i}\delta_{n_x,n_y}(t)]v'(\hat{z}) = \sqrt{\frac{(\omega_1\omega_2)^{1/2}}{2\hbar\pi L(\alpha_1\alpha_2)^{1/2}}}\frac{1}{\sqrt{2^{n_x+n_y}n_x!n_y!}} \times H_{n_x}\left(\sqrt{\frac{\omega_1}{2\alpha_1\hbar}}\hat{X}\right)H_{n_y}\left(\sqrt{\frac{\omega_2}{2\alpha_2\hbar}}\hat{Y}\right)$$

$$\times \exp\left[-\frac{1}{2\alpha_1\hbar}\left(\frac{\omega_1}{2}+\mathrm{i}\beta_1\right)\hat{X}^2-\frac{1}{2\alpha_2\hbar}\left(\frac{\omega_2}{2}+\mathrm{i}\beta_2\right)\hat{Y}^2\right]$$
$$\times \exp(\mathrm{i}k'_q\hat{z})\exp\left[-\mathrm{i}\omega_1\left(n_x+\frac{1}{2}\right)\int_0^t\frac{1}{2\alpha_1(t')m^*(t')}\,\mathrm{d}t'\right.$$
$$\left.-\,\mathrm{i}\omega_2\left(n_y+\frac{1}{2}\right)\int_0^t\frac{1}{2\alpha_2(t')m^*(t')}\,\mathrm{d}t'\right], \tag{3.40}$$

where

$$\begin{pmatrix}\hat{X}\\ \hat{Y}\end{pmatrix}=\begin{pmatrix}\cos\frac{\varphi}{2} & -\sin\frac{\varphi}{2}\\ \sin\frac{\varphi}{2} & \cos\frac{\varphi}{2}\end{pmatrix}\begin{pmatrix}\hat{x}\\ \hat{y}\end{pmatrix}. \tag{3.41}$$

In the calculation of equation (3.40), we used the identity given by

$$\exp\left[\varphi\left(\hat{y}\frac{\partial}{\partial\hat{x}}-\hat{x}\frac{\partial}{\partial\hat{y}}\right)\right]f(\hat{x},\hat{y})=f(\hat{x}\cos\varphi+\hat{y}\sin\varphi,-\hat{x}\sin\varphi+\hat{y}\cos\varphi). \tag{3.42}$$

4. Summary and discussion

We used the dynamical invariant method to obtain quantum solutions of an electron moving under vector potential with time-dependent effective mass. With the choice of the Landau gauge, the Schrödinger equation is reduced to that of a one-dimensional time-dependent harmonic oscillator. The invariant operator can be expressed in terms of classical particle solutions, x_{p} and $p_{x,\mathrm{p}}$, of the equation of motion.

We also derived quantum solutions for the choice of the symmetric gauge given in equation (3.1). If we consider the revolving motion of an election in the x–y plane under this gauge, the excitations and frequencies of $\hat{x}$ and $\hat{y}$ components are naturally the same as each other:

$$n_x=n_y\equiv n_0, \tag{4.1}$$
$$\omega_1=\omega_2\equiv\omega_0. \tag{4.2}$$

Then, equation (3.34) can be rewritten as

$$\lambda_{n_x,n_y}=2\hbar\omega_0\left(n_0+\tfrac{1}{2}\right). \tag{4.3}$$

By comparing equations (2.17) and (3.12), we can confirm that the frequency in the symmetric gauge is half of that in the Landau gauge:

$$\omega_0=\tfrac{1}{2}\omega. \tag{4.4}$$

Therefore, equation (4.3) exactly reduces to equation (2.36) which is obtained under the Landau gauge.

References

[1] Dantas C M A, Pedrosa I A and Baseia B 1992 *Phys. Rev.* A **45** 1320
[2] Ji J-Y, Kim J K and Kim S P 1995 *Phys. Rev.* A **51** 4268
[3] Marchiolli M A and Mizrahi S S 1997 *J. Phys. A: Math. Gen.* **30** 2619
[4] Maamache M, Provost J-P and Vallée G 1999 *Phys. Rev.* A **59** 1777
[5] Dekar L, Chetouani L and Hammann T F 1999 *Phys. Rev.* A **59** 107
[6] Chetouani L, Dekar L and Hammann T F 1995 *Phys. Rev.* A **52** 82
[7] Plastino A R, Rigo A, Casas M, Garcias F and Plastino A 1999 *Phys. Rev.* A **60** 4318
[8] Moiseyev N and Lefebvre R 2001 *Phys. Rev.* A **64** 052711
[9] Pistol M E 1999 *Phys. Rev.* B **60** 14269

[10] Colegrave R K and Abdalla M S 1981 *Opt. Acta* **28** 495
[11] Colegrave R K and Abdalla M S 1982 *J. Phys. A: Math. Gen.* **15** 1549
[12] Remaud B and Hernandez E S 1980 *J. Phys. A: Math. Gen.* **13** 2013
[13] Um C I, Kim I H, Yeon K H, George T F and Pandey L N 1997 *J. Phys. A: Math. Gen.* **30** 2545
[14] Um C I, Choi J R and Yeon K H 2001 *J. Korean Phys. Soc.* **38** 452
[15] Johnson P D, Valla T, Fedorov A V, Yusof Z, Wells B O, Li Q, Moodenbaugh A R, Gu G D, Koshizuka N, Kendziora C, Jian S and Hinks D G 2001 *Phys. Rev. Lett.* **87** 177007
[16] Im H, Klipstein P C, Grey R and Hill G 2000 *Phys. Rev.* B **62** 11076
[17] Kleimann P, Semmache B and Berre M L 1998 *Phys. Rev.* B **57** 8966
[18] Vretenar D, Nikšić T and Ring P 2002 *Phys. Rev.* C **65** 024321
[19] Gevorkian Z S and Lozovik Y E 1988 *J. Phys. C: Solid State Phys.* **21** 2181
[20] Lewis H R Jr 1967 *Phys. Rev. Lett.* **18** 510
[21] Lewis H R Jr 1968 *J. Math. Phys.* **9** 1976
[22] Lewis H R Jr and Riesenfeld W B 1969 *J. Math. Phys.* **10** 1458
[23] Dittrich T, Hanggi P, Ingold G-L, Kramer B, Schön G and Zwerger W *Quantum Transport and Dissipation* (Weinheim: Wiley–VCH)
[24] Lewis H R, Bates J W and Finn J M 1996 *Phys. Lett.* A **215** 160
[25] Lee J Y, Liu K L and Lo C F 1998 *Phys. Rev.* A **58** 3433
[26] Skála L, Čížek J, Weniger E J and Zamzstil J 1999 *Phys. Rev.* A **59** 102
[27] Hagedorn G A, Loss M and Slawny J 1986 *J. Phys. A: Math. Gen.* **19** 521
[28] Erdély A 1953 *Higher Transcendental Functions* vol 2 (New York: McGraw-Hill)